The Family of Rectifying Homographies Generated by the Circular Points

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Last revision: June 2008

Given a plane π, a planar homography $\mathtt{H}_\pi$ maps any point $\mathbf{X}_\pi \in \pi$ onto the image plane as $\mathbf{x}_\pi \sim \mathtt{H}_\pi \mathbf{X}_\pi$, where points are expressed in homogeneous coordinates, and the symbol "$\sim$" denotes equality up to a (complex) scale factor. In particular, the homography maps the canonical circular points on π, $\mathbf{I}_\pi \doteq [1 \;\; i \;\; 0]^\top$ and $\mathbf{J}_\pi = \mathrm{conj}(\mathbf{I}_\pi)$, respectively as

$$\mathbf{i}_\pi \sim \mathtt{H}_\pi \mathbf{I}_\pi \tag{1}$$

and $\mathbf{j}_\pi = \mathrm{conj}(\mathbf{i}_\pi)$. A homography $\mathtt{H}_\mathrm{R}$ is said to *rectify the image of π in a metric sense* if and only if $\mathtt{H}_\pi^{-1} = \mathtt{H}_\mathrm{R}\mathtt{H}_\mathrm{M}$ for some 4-dof metric (similarity) transformation of the plane $\mathtt{H}_\mathrm{M}$ [1]. Hence, to rectify a plane in a metric sense, only four out of the eight dofs of $\mathtt{H}_\pi$ are required, so that there exist ∞^4 possible rectifying homographies compatible with $\mathtt{H}_\pi$. The fundamental property any rectifying homography must meet is to act as the inverse of $\mathtt{H}_\pi$ at circular points:

$$\mathtt{H}_\mathrm{R}\, \mathbf{i}_\pi \sim \mathbf{I}_\pi \;\; ; \tag{2}$$

this is because the circular points pair is invariant to metric transformations.

A general expression for the 4-parameter family of rectifying homographies is now derived, where $\mathtt{H}_\mathrm{R}$ will be expressed in terms of the four free parameters and the four fixed values encoding circular point information. Let us start by recalling that the original planar homography can be decomposed as the product of a projective transformation (2 dofs), $\mathtt{H}_\mathrm{P}$, and an affine one (6 dofs), $\mathtt{H}_\mathrm{A}$:

$$\mathtt{H}_\pi \sim \mathtt{H}_\mathrm{P}\, \mathtt{H}_\mathrm{A} \;\; . \tag{3}$$

The latter transformation can be written as

$$\mathtt{H}_\mathrm{A} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \;\; , \tag{4}$$

where the invertible matrix $\mathtt{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ controls scale, rotation, and skew along two orthogonal directions, and $\mathbf{t} = \begin{bmatrix} r \\ s \end{bmatrix}$ controls rigid translations. Affine transformations can move circular points, but they cannot make them leave the line at infinity:

$$\mathtt{H}_\mathrm{A}\mathbf{I}_\pi = \begin{bmatrix} a + ib \\ c + id \\ 0 \end{bmatrix} \;\; . \tag{5}$$

Eq. 5 also shows that $\mathbf{I}_\pi$ is not changed by a translation: This is a consequence of the abovementioned invariance of circular points with respect to general similarity transformations. The 2-dof projective transformation brings the ideal point $\mathtt{H}_\mathrm{A}\mathbf{I}_\pi$ to the finite point $\mathbf{i}_\pi \doteq [\alpha + i\beta \;\; \gamma + i\delta \;\; 1]^\top$ of the image plane. Its inverse can be written as

$$\mathtt{H}_\mathrm{P}^{-1} = \begin{bmatrix} \mathtt{I}_{2\times 2} & \mathbf{0} \\ \multicolumn{2}{c}{\mathbf{l}_{\infty\pi}^\top} \end{bmatrix} \;\; , \tag{6}$$

where $\mathbf{l}_{\infty\pi} \doteq \dfrac{1}{2i}(\mathbf{i}_\pi \times \mathbf{j}_\pi) = [\delta \;\; -\beta \;\; -(\alpha\delta - \beta\gamma)]^\top$ is the vanishing line of π, passing through both the imaged circular points. (In order for $\mathtt{H}_\mathrm{P}$ to be invertible, $\det(\mathtt{H}_\mathrm{p})^{-1} = (\alpha\delta - \beta\gamma)$ must be nonzero.) The effect of $\mathtt{H}_\mathrm{P}^{-1}$ is to bring back the imaged circular point onto the line at infinity, by changing to 0 the third component without touching the first two:

$$\mathtt{H}_\mathrm{P}^{-1}\mathbf{i}_\pi = \begin{bmatrix} \alpha + i\beta \\ \gamma + i\delta \\ 0 \end{bmatrix} . \tag{7}$$

Now, combining Eqs. 1 and 3 yields $\mathtt{H}_\mathrm{P}^{-1}\mathbf{i}_\pi \sim \mathtt{H}_\mathrm{A}\mathbf{I}_\pi$, so that for every nonzero (complex) value $\lambda = p + iq$ there must exist an affine transformation compatible with the imaged circular points:

$$\begin{bmatrix} a + ib \\ c + id \\ 0 \end{bmatrix} = (p + iq) \begin{bmatrix} \alpha + i\beta \\ \gamma + i\delta \\ 0 \end{bmatrix} . \tag{8}$$

Therefore, the required ∞^4 rectifying homographies are obtained by all possible choices of the parameter 4-tuple (p, q, r, s), by solving Eq. 8 for (a, b, c, d), thus finding the expression of all the affine transformations compatible with the imaged circular points:

$$\mathtt{H}_\mathrm{A}(p, q, r, s) = \begin{bmatrix} p\alpha - q\beta & q\alpha + p\beta & r \\ p\gamma - q\delta & q\gamma + p\delta & s \\ 0 & 0 & 1 \end{bmatrix} , \tag{9}$$

where $\det(\mathtt{H}_\mathrm{A}(p, q, r, s)) = \det(\mathtt{A}(p, q)) = (p^2 + q^2)(\alpha\delta - \beta\gamma)$, and finally obtaining, from Eqs. 1 through 3:

$$\mathtt{H}_\mathrm{R}(p, q, r, s) \sim \mathtt{H}_\mathrm{A}^{-1}(p, q, r, s)\, \mathtt{H}_\mathrm{P}^{-1} . \tag{10}$$

A particularly expressive form for the rectifying homography is obtained for $r = s = 0$:

$$\mathtt{H}_\mathrm{R}(p, q, 0, 0) \sim \begin{bmatrix} \mathtt{A}^{-1}(p, q) & \mathbf{0} \\ \multicolumn{2}{c}{\mathbf{l}_{\infty\pi}^\top} \end{bmatrix} . \tag{11}$$

It is worth noting that the entries of the last row of this homography are nothing but the components of the vanishing line of π. The above expression generalizes the particular solution reported in [2], which is obtained for $p = \frac{\delta}{\gamma^2+\delta^2}(\alpha\delta - \beta\gamma)^{-1}$, $q = \frac{\gamma}{\gamma^2+\delta^2}(\alpha\delta - \beta\gamma)^{-1}$, and also allows the derivation of the following simple rectifying transformation:

$$\mathtt{H}_\mathrm{R}((\alpha\delta - \beta\gamma)^{-1}, 0, 0, 0) = \begin{bmatrix} \delta & -\beta & 0 \\ -\gamma & \alpha & 0 \\ \delta & -\beta & -(\alpha\delta - \beta\gamma) \end{bmatrix} . \tag{12}$$

References

[1] R.I. Hartley and A. Zisserman. *Multiple View Geometry in Computer Vision.* Cambridge University Press, 2nd ed. 2004.

[2] D. Liebowitz and A. Zisserman, Metric rectification for perspective images of planes. In *Proc. IEEE Conference on Computer Vision and Pattern Recognition*, 1998.